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NOTES ON LINEAR INEQUALITIES, II: THE GEOMETRY OF SOLVABILITY AND DUALITY IN LINEAR PROGRAMMING

by

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SYSTEMS RESEARCH GROUP
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Abstract

Solvability and boundedness criteria for dual linear programming problems are given in terms of the problem data and the intersections of the nonnegative orthant with certain complementary orthogonal subspaces.

Introduction

The duality theorem of linear programming [1/] relates two linear extremum problems in terms of solvability, boundedness and equality of functional values. The classical theory of Lagrange multipliers admits extensions to some special nonlinear cases [2/] as well as interpretations of duality in the context of applications. [3/]

Tucker, in [16], showed duality--in the linear case-- to follow from elementary geometric considerations of complementary orthogonality of manifolds corresponding to the dual problems. [4/]

In this paper we follow Tucker's approach and supplement his results, in [16], by an alternative theorem for dual programs, theorem 4 below, and by a characterization of all duality situations in terms of the geometrical configurations of certain manifolds associated with the data and the data itself, theorem 6 below.

None of our results seem to be essentially new; yet our efforts may be justified for pedagogical reasons.

1/ Conjectured by Von Neumann (e.g. [7], p. 23) and proved by Gale, Kuhn and Tucker [8]. For the extended form discussed here, see Charnes and Cooper [3].

2/ Notably Kuhn and Tucker [11]

3/ E.g. [7], pp. 19-22.

4/ A similar approach was used in [1] to develop, in a unified manner, the main theorems of linear inequalities.

0. Notations

In this note we use the same notations as in [1]. Recall in particular that:

$\mathcal{F}$ is an arbitrary ordered field

E^n is the n-dimensional vector space over $\mathcal{F}$

$C\{f_1, \ldots, f_k\}$ is the cone spanned by the vectors $f_1, \ldots, f_k$ in E^n

A is an $m \times n$ matrix over $\mathcal{F}$

$N(A)$ is the null space of A in E^n

$R(A^T)$ is the range space of A^T in E^n.

In addition let A^+ denote the generalized inverse of A, e.g. [14] and [2].

1. Lemma: Let L be an arbitrary subspace of E^n. Then the following are equivalent:

(i) $\{x + L\} \cap E_+^n \neq \phi$ for all $x \in E^n$

(ii) $L \cap \operatorname{int} E_+^n \neq \phi$

(iii) $\{x + L\} \cap \operatorname{int} E_+^n \neq \phi$ for all $x \in E^n$

Proof:

(i) ==> (ii)

Suppose (ii) is false. This is possible only in two cases:

Case A: $L \cap \operatorname{bdry} E_+^n = C\{e_1, \ldots, e_p\}$ $\quad 1 \leq p \leq n - \dim L$

Case B: $L \cap E_+^n = \{\theta\}$

We will now show (i) to be false by producing, in each case, a vector x_o such that $\{x_o + L\} \cap E_+^n = \phi$.

Case A: Let x_o be any vector in $L^\perp \cap \operatorname{int} C\{-e_{p+1}, \ldots, -e_n\}$, a set which is nonempty by [1], corollary 5.

Case B: Let x_o be any vector in $L^\perp \cap \operatorname{int}\{-E_+^n\}$; the latter set is nonempty by [1], corollary 2.

(ii) ==> (iii)

Let $\ell \in L \cap \operatorname{int} E^n_+$. Then for all $x \in E^n$ and scalars λ satisfying

$$\lambda > \max_{x_i < 0} \frac{|x_i|}{\ell_i}$$

we have: $$x + \lambda \ell \in \operatorname{int} E^n_+$$

(iii) ==> (i) Obvious

∎

2. Corollary: Let A be an arbitrary $m \times n$ matrix over $\mathcal{F}$. Then the following are equivalent:

(i) $Ax = b$, $x \geqq \theta$ is solvable for all $b \in R(A)$

(ii) $Ax = \theta$, $x > \theta$ is solvable

(iii) $Ax = b$, $x > \theta$ is solvable for all $b \in R(A)$.

Proof:

The solutions of $Ax = b$, when solvable, form the manifold: $A^+b + N(A)$, e. g. [14] . Now (i), (ii) and (iii) are the corresponding parts in lemma 1 with $x = A^+b$ and $L = N(A)$.

∎

3. Corollary: Let A be an arbitrary $m \times n$ matrix over $\mathcal{F}$. Then the following are equivalent:

(i) $A^T w \geqq c$ is solvable for all $c \in E^n$

(ii) $A^T w > 0$ is solvable

(iii) $A^T w > c$ is solvable for all $c \in E^n$.

Proof: In lemma 1 let $L = R(A^T)$, $x = -c$.

∎

4. Theorem: Let A be an arbitrary $m \times n$ matrix over $\mathcal{F}$. Consider the system of equations and inequalities:

I) $Ax = b$, $x \geqq \theta$ I') $Ax = \theta$, $x \geqq \theta$

II) $A^T w \geqq c$ II') $A^T w \geqq \theta$

Then:

a) I is solvable for all $b \in R(A)$ if and only if II' does not have solutions w with nonnegative nonzero vectors $A^T w$.

b) II is solvable for all $c \in E^n$ if and only if I' does not have nonnegative nonzero solutions.

c) If II' has solutions w with $A^T w \geq \theta$ then I is solvable if and only if $A^T w \geq \theta \implies (b, w) \geq 0$.

d) If I' has solutions x with $x \geq \theta$ then II is solvable if and only if $Ax = \theta$, $x \geq \theta \implies (c, x) \leq 0$.

Proof:

a) By corollary 2 it follows that I is solvable for all $b \in R(A)$ if and only if $N(A) \cap \text{int}\, E^n_+ \neq \phi$. By [1], corollary 2, the latter condition is equivalent to $R(A^T) \cap E^n_+ = \{\theta\}$.

b) By corollary 3, II is solvable for all $c \in E^n$ if and only if $R(A^T) \cap \text{int}\, E^n_+ \neq \phi$. This is equivalent, by [1], corollary 2, to: $N(A) \cap E^n_+ = \{0\}$.

c) This is the well-known Farkas' lemma, e.g. [15].

d) This is theorem 1 in Ky Fan [6]. ∎

Remarks

a) Theorem 4 is a collection of classical results in a setup which is completely analogous to "Fredholm's Alternative" theorem for linear equations, [5]. Solvability relations between linear inequalities and equations were studied by Motzkin [13], Kuhn [10] and generalized by Ky Fan to the case of complex normed linear spaces [6]. For a use of Fredholm's theorem to prove the main theorems of linear inequalities see [1].

b) For $b \in R(A)$, part c can be rewritten as:

c') If $b \in R(A)$ and II' has solutions w with $A^T w \geq \theta$ then I is solvable if and only if

$$A^T w \geq \theta \implies (A^T w,\ A^+ b) \geq 0 .$$

This follows from the fact that AA^+ is the perpendicular projection on $R(A)$, e.g. [2].

5. Let A be an arbitrary $m \times n$ matrix over $\mathcal{F}$, $b \in E^m$ and $c \in E^n$. Let

$$S = \{x \in E^n : Ax = b,\ x \geqq \theta\} \qquad I_1 = \sup_{x \in S} (c, x)$$

$$T = \{w \in E^m : A^T w \geqq c\} \qquad I_2 = \inf_{w \in T} (b, w)$$

The duality theorem of linear programming relates the problem of solving for I_1, the <u>primal</u> problem, to that of solving for I_2, the <u>dual</u> problem.

The duality theorem states indeed that there are four mutually exclusive cases:

Case A: $S \neq \phi$, $T \neq \phi$, $I_1 = I_2$

Case B: $S = \phi$, $T \neq \phi$, $I_2 = -\infty$

Case C: $S \neq \phi$, $T = \phi$, $I_1 = \infty$

Case D: $S = \phi$, $T = \phi$

Conjectured by Von Neumann, e.g. [7], p. 23, and proved by Gale, Kuhn and Tucker [8], this theorem was extended to some nonlinear situations, the most general being that of Charnes, Cooper and Kortanek [4].

We will now elaborate on the four cases given above. In terms of the data $\}A, b, c\}$, and more specifically of the configurations of $N(A)$ and $R(A^T)$ with respect to E^n_+, we give below conditions for the attainment of each of the above cases.

6. Theorem:

Let A be an arbitrary $m \times n$ matrix over [illegible], $b \in R(A)$ in E^m, $c \in E^n$ and let S, T, I_1 and I_2 be as above. Then there are eight mutually exclusive cases, tabulated below:

Assumptions			Conclusions		
case	intersection with E_+^n	conditions on b, c	S	T	I_1, I_2
1	$N(A) \cap E_+^n = \{\theta\}$	$A^T w \geq \theta \implies (A^T w, A^+ b) \geqq 0$	non-empty	non-empty for all $c \in E^n$	$I_1 = I_2$
2		$A^T w \geq \theta$ and $(A^T w, A^+ b) < 0$ for some w	empty	"	$I_2 = \infty$
3	$R(A^T) \cap E_+^n = \{\theta\}$	$Ax = \theta,\ x \geq \theta \implies (c, x) \leq 0$	non-empty for all $b \in R(A)$	non-empty	$I_1 = I_2$
4		$Ax = \theta,\ x \geq \theta$ and $(c, x) > 0$ for some x	"	empty	$I_1 = \infty$
5	$N(A) \cap \text{bdry}\ E_+^n = C\{e_1 \ldots e_p\}$ $1 \leq p \leq \dim N(A)$ $R(A^T) \cap \text{int}\ C\{e_{p+1}, \ldots, e_n\} \neq \phi$	$A^T w \geq \theta \implies (A^T w, A^+ b) \geqq 0$ and $Ax = \theta,\ x \geq \theta \implies (c, x) \leq 0$	non-empty	non-empty	$I_1 = I_2$
6		$A^T w \geq \theta \implies (A^T w, A^+ b) \geqq 0$ but $Ax = \theta,\ x \geq \theta$ and $(c, x) > 0$ for some x	"	empty	$I_1 = \infty$
7		$A^T w \geq \theta$ and $(A^T w, A^+ b) < 0$ for some w but $Ax = \theta,\ x \geq \theta \implies (c, x) \leq 0$	empty	non-empty	$I_2 = -\infty$
8		$A^T w \geq \theta$ and $(A^T w, A^+ b) < 0$ for some w and $Ax = \theta,\ x \geq \theta$ and $(c, x) > 0$ for some x	"	empty	

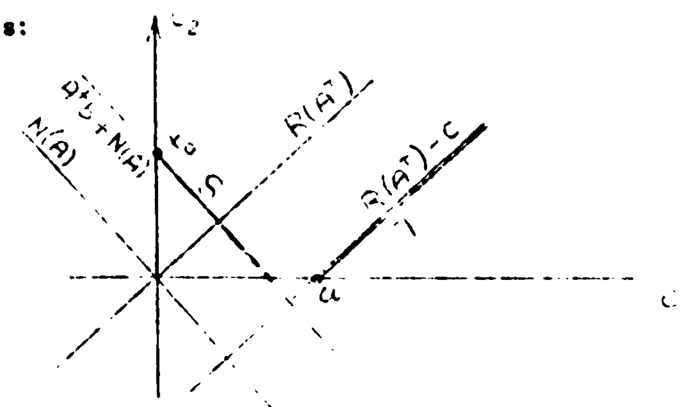
Proof:

The cases 1, . . . , 8 are clearly mutually exclusive. In each case theorem 4 is used to draw the conclusions regarding the sets S and T. Then the duality theorem of linear programming is used to obtain I_1 and I_2.

□

Remarks:

a) The above 8 cases can be visualized geometrically in a manner which helps to clarify the concept of duality. Thus in the 2-dimensional case where A is a 1×2 matrix, $\dim R(A^T) = \dim N(A) = 1$, the first case appears as follows:



The other cases are drawn in a similar manner. Furthermore, by the "complementary slackness" property, it is now easy to identify optimal points. Thus x_o is the optimal solution of the primal problem and $a = A^T w_o - c$ where w_o is the optimal solution of the dual problem, e.g., Tucker [16], p. 15.

b) Theorem 6 combines well-known solvability theorems, e.g., Tucker [16] and Charnes-Cooper [3], p. 214, and the duality theorem of linear programming to characterize the duality situations in terms of the data $\{A, b, c\}$.

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